ON THE STABILITY OF THE EQUILIBRIUM OF CONSERVATIVE SYSTEMS WITH AN INFINITE NUMBER OF DEGREES OF FREEDOM

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By means of examples involving some simple systems with an infinite number of degrees of freedom, an analysis is made of the possibility of the existence of a direct method of proof of stability with the aid of Liapunov's functional of the increment of the total energy of a system. The direct method of proof is used to establish the stability of equilibrium defined in the linear theory of elasticity.

1. The basis of the direct proof of the classical theorem on the stability of the equilibrium of a conservative system with a finite number of degrees of freedom [1], is the following property of a continuous function of a finite number of variables: the lower boundary, of the difference between the values of a function on a sphere of sufficiently small radius with center at a point of a strict relative minimum, and the minimum value, is positive. This property is the essential point in the proof of the general theorem on stability [2,3]. In somewhat stronger form, it plays a role in the theory of the direct method of Liapunov [4] (definition (4.1)). In order to avoid the introduction of a new terminology, we will call this property the definite positiveness of the increment of the function. When one goes over from discrete to continuous systems, which corresponds to the shift from functions to functionals, then there arise the following questions:

(1) For what metrics of the function space is the property of the definite positiveness of the increment of the functional preserved?

(2) To what extent does the energy integral give a complete picture of the stability as compared to that given by the results that follow from the direct solution of Cauchy's problem?

2. We will consider the problem on the stability of the equilibrium of a uniform free string stretched between two fixed points. The plane motion is described by the equations

$$\mu u_{tt} = T_1 u_{xx} \qquad (0 \leqslant x \leqslant l,\ t \geqslant 0), \qquad u(0, t) = 0, \qquad u(l, t) = 0 \tag{2.1}$$

Here μ is the linear density, and T_1 is the tension of the string. The zero initial conditions correspond to the equilibrium

$$u(x, t) \equiv 0 \tag{2.2}$$

The total energy

$$H = \frac{1}{2}\int_0^l \mu u_t^2\, dx + \frac{1}{2}\int_0^l T_1 u_x^2\, dx \tag{2.3}$$

remains constant along the motion, and is continuous with respect to the metric

$$\rho_1 = \sup_x |u| + \sup_x |u_x| + \sup_x |u_t| \qquad (0 \leqslant x \leqslant l) \tag{2.4}$$

Because of the inequalities

$$\int_0^l u_x^2\, dx \geqslant \frac{\pi^2}{l^2}\int_0^l u^2\, dx, \qquad \int_0^l u_x^2\, dx > \frac{4}{l}(\sup_x |u|)^2 \tag{2.5}$$

the total energy is positive definite with respect to the metrics ρ_2, and ρ_3

$$\rho_2 = \left\{\int_0^l (u^2 + u_x^2 + u_t^2)\, dx\right\}^{1/2}, \qquad \rho_3 = \sup_x |u| \quad (0 \leqslant x \leqslant l) \tag{2.6}$$

This implies [4,5] the stability of the equilibrium (2.2) with respect to the metrics ρ_1, ρ_2 and ρ_1, ρ_3.

As is known [4], the solution of Cauchy's problem is given by the formula

$$u(x, t) = \frac{1}{2}\left[u\ (x - at) + u_0(x + at) + \frac{1}{a}\int_{x-at}^{x+at} v_0(\xi)\, d\xi\right] \qquad \left(a = \sqrt{\frac{T_1}{\mu}}\right) \tag{2.7}$$

where u_0 and v_0 are functions obtained by an odd, periodic extension (of period $2l$) to the entire real axis of the initial values of the deviations and velocities of the points of the string.

From (2.7) one can easily obtain the stability of equilibrium (2.2) with respect to the metrics ρ_4 and ρ_3

$$\rho_4 = \sup_x |u| + \sup_x |u_t| \qquad (0 \leqslant x \leqslant l) \tag{2.8}$$

and also the stability of (2.2) with respect to ρ_1.

The total energy H is not positive definite with respect to the metric ρ_1 or with respect to the "statistical" metric

$$\rho_{1s} = \sup_x |u| + \sup_x |u_x| \qquad (0 \leqslant x \leqslant l) \tag{2.9}$$

This example shows that the energy integral gives only a partial picture of the equilibrium stability of a system.

3. If the potential energy of the system is a functional of a function of one variable, then the fulfillment of the hypotheses of Osgood's [7] theorem (which are somewhat stronger than those that are sufficient for the existence a strong minimum of the potential energy at the position of equilibrium) will guarantee the definite positiveness, with respect to the metric ρ_3, of the increment of the total energy of the system, and, hence, the equilibrium stability with respect to the metrics ρ_1, ρ_3.

It was in this manner that Kneser proved the equilibrium stability of a string fastened at both ends in a gravitational field [8]. Born [9] used analogous arguments in his study of the equilibrium stability of rods.

But for systems, whose potential energy depends on functions of more than one variable, and on their derivatives of order not greater than one, one cannot expect [10, 11] the positive definiteness of the energy increment for a metric of type ρ_3.

Let us consider, for example, the equilibrium stability of a membrane, clamped along the periphery, under the action of a constant transverse load. The increment of the potential energy of the membrane has the form

$$U = \frac{1}{2} \iint_D T_2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy, \qquad u|_\Gamma = 0 \tag{3.1}$$

if one takes account of the membrane's deviation from the position of equilibrium. Here, T_2 is the stress in the membrane.

Hadamard [10] has shown that this functional is not positive definite relative to the metric

$$\rho_5 = \sup_{x,y} |u|, \quad (x, y) \in D \tag{3.2}$$

However, an analog of the first inequality of (2.5) is valid [11]:

$$\iint_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \geqslant C_1 \iint_D u^2\, dx\, dy \tag{3.3}$$

Here C_1 is a fixed positive constant independent of u. On the basis of this, we can draw certain conclusions on the equilibrium stability with respect to the metrics ρ_6, ρ_7:

$$\rho_6 = \sup_{x,y} |u| + \sup_{x,y} |u_x| + \sup_{x,y} |u_y| + \sup_{x,y} |u_t|, \qquad (x, y) \in D \tag{3.4}$$

$$\rho_7 = \left\{\iint_D (u^2 + u_x^2 + u_y^2 + u_t^2)\, dx\, dy\right\}^{1/2} \tag{3.5}$$

4. We consider next the question on the equilibrium stability in a field of the mass forces (constant in time) of an elastic body in the linear theory of elasticity. Suppose that on the part σ' of the surface of a body there is given a displacement $\mathbf{u}^\circ$, independent of time

$$\mathbf{u} = \mathbf{u}^\circ(x) \qquad (x \in \sigma') \tag{4.1}$$

on the part σ'', the tension is $\mathbf{t}_n^\circ$

$$\mathbf{t}_n = \mathbf{t}_n^\circ(x) \qquad (x \in \sigma'') \tag{4.2}$$

while on the part σ''', the contact condition is

$$u_n = u_n^\circ(x), \qquad t_{ns} = 0 \qquad (x \in \sigma''') \tag{4.3}$$

The case when one of the parts σ'' and σ''' is absent, is not excluded from consideration.

The increment of the potential energy of a body in the investigation of its displacement from the position of equilibrium, has the form

$$V = \frac{1}{2}\int_\omega c_{ijkl} u_{i,j} u_{k,l}\, d\omega$$
$$\mathbf{u} = 0 \quad (x \in \sigma'), \qquad \mathbf{t}_n = 0 \quad (x \in \sigma''); \qquad u_n = 0, \quad t_{ns} = 0 \quad (x \in \sigma''') \tag{4.4}$$

Here, c_{ijkl} is the tensor of the elastic constants. The repeated indices indicate summation from 1 to 3. The inequality [13]

$$\int_\omega u_{i,j} u_{i,j}\, d\omega \geqslant C_2 \int_\omega u_i u_i\, d\omega \tag{4.5}$$

$$\int_\omega c_{ijkl} u_{i,j} u_{k,l}\, d\omega \geqslant C_3 \int_\omega u_{i,j} u_{i,j}\, d\omega \qquad \text{(Korn's inequality)} \tag{4.6}$$

where C_2 and C_3 are fixed positive constants that do not depend on $\mathbf{u}$, implies the equilibrium stability with respect to the metrics ρ_8, ρ_9:

$$\rho_8 = \sup_x |\mathbf{u}| + \sup_x \sqrt{u_{i,j} u_{i,j}} + \sup_x |\mathbf{u}_t| \qquad (x \in \omega) \tag{4.7}$$

$$\rho_9 = \left\{ \iint\limits_{\omega} (u^2 + u_{i,j}\ u_{i,j}\ + u_t^2)\, d\omega \right\}^{1/2} \tag{4.8}$$

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